# Comparing Subjective Randomness with Poisson Distribution

Author: Valentina Mkhitaryan

*BS in Data Science*
*American University of Armenia*

Supervisor: Dr. Tadamasa Sawada

***Abstract*— Past research has successfully demonstrated that humans are in fact able to perceive and respond to randomness in the environment. However, it is also well known that humans' understanding of random events is biased – they tend to see regularities in truly-random data. This tendency is referred to as the "cluster illusion". This study focuses on two-dimensional point patterns to analyze the statistical properties of the distribution of subjective randomness by comparing it to the properties of the Poisson distribution. To do so, past research studies are reviewed and 2D dot patterns are generated utilizing different algorithms used in past behavioral experiments. The possibility of modeling subjective randomness with the Image Pyramid as well as its limitations are discussed.**

***Keywords*—Randomness, Point Process, Visual Perception, Cluster Illusion, Poisson Distribution, Image Pyramid**

## I. Introduction

The randomness of a process is often interpreted as how unpredictable each individual event within the series is [1]. On the other hand, the randomness of a process is typically evaluated based on its output, which is expected to exhibit no discernible patterns [2]. These two definitions of randomness are referred to as "primary" and "secondary" randomness [3, 2]. Which of these two definitions do humans use to evaluate the randomness of the events – do they rely on the method used to generate the stimuli, or the properties of the output? And based on which properties of the latter do they make their judgements about randomness? For years, vast research has been conducted to answer these questions about humans' perception of randomness.

Randomness is a fundamental aspect of many natural and man-made systems. Studying human perception of randomness and the biases that are involved in it can help us identify and correct for the resulting errors in human decision-making. Furthermore, it can have broader implications for developing deeper insights into human cognition and the nature of randomness itself. There are also many practical applications of the study of randomness, for example in risk management, statistical analysis, gambling, and other fields that rely on accurate prediction of uncertain outcomes. Another example is behavioral microeconomics, where understanding how people perceive randomness and make decisions based on that perception can help develop better agent-based economic models, as the actions of individual human agents can have big influence on the behavior of the model in the bigger picture.

It has been shown that humans possess the ability to detect and respond to randomness [4]. However, past studies have demonstrated that humans have a biased perception of randomness in data, as they tend to see patterns where none really exist [5]. Data is perceived as more random when events are evenly distributed, compared to truly-random data. This phenomenon is known as the "cluster illusion". Having a grasp of this illusion is significant for designing better systems that take the biases of human perception of randomness into account. The purpose of this study is to compare the two-dimensional distributions of true- and subjective- randomness by analyzing them using the Poisson distribution.

In the past, numerous studies have explored the human ability to perceive and respond to randomness in one-dimensional (1D) and two-dimensional (2D) patterns. The main consistent discovery from research on the generation and perception of randomness in binary sequences (including 2D grids) is that humans tend to associate randomness with an excess of alterations between different types of symbols [2]. Wilke et al. [6] view this tendency of perceiving clusters in truly-random patterns as evidence that the human mind has evolved to expect resources (such as food and mates) in the natural environment in clumps or patches.

In his Introduction to Probability Theory, Feller remarks: "To the untrained eye, randomness appears as regularity or tendency to cluster" [7, 2]. He describes the famous example of the flying-bomb attacks on London during the Second World War, which was widely believed to be non-random, as some parts of the city were hit repeatedly while some others were not hit at all. To test this hypothesis, the area of the city was divided into small square regions of equal area: as it turned out, the resulting distribution of the number of attacks per region closely followed the Poisson distribution, which would be expected if the attacks were in fact random [5].

In their study, Kahneman and Tversky [8] explored the phenomenon of human perception of clustering and randomness in 1D patterns (for example, binary sequences of "Heads" or "Tails" outcomes from tossing a coin repeatedly) using the local representativeness heuristic – judging the probability of an event based on how representative it is of the entire population; that is, the descriptive characteristics of the parent population are expected to be present locally in the smaller samples. However, the locally representative samples contain too many alternations and too few clusters, as compared to chance expectations. As a result, the locally representative samples, which humans tend to identify as more random, appear to be more regular than expected by truly-random processes.

A similar conclusion was reached by Falk and Konold [2], who suggested that the human judgement regarding the randomness of binary sequences relied on the implicit encoding of the patterns. That is, the degree of randomness is based on how "hard" it is to encode the sequence using the relative frequencies of typical subsequences. On the one hand, this leads to biases and errors in the judgement of randomness, as sequences with an excessive number of changes between symbols are perceived as more random than their actual entropy and algorithmic randomness suggest. However, on the other hand, it also indicates that humans' intuitive understanding of the concept of randomness is consistent with the mathematical view of randomness as maximal complexity.

Some other studies focused on 2D dot patterns instead of 1D sequences. Matsuda and Kaneko [9] investigated the human perception of order and disorder in discrete 2D space using dot patterns aligned on a square grid (4x4 square black areas, 8 of them containing white dots and the other 8 being uniformly black). The participants were briefly shown two patterns at a time, and after the patterns had been hidden from view, they were asked to report which one was the more disorderly of the two. The results were consistent with those of the previous studies focusing on discrete 1D space: the patterns with high gathering index and high repetition index tended to be rated as more orderly (less random) by the participants. Moreover, the high correlation between the two indices indicated that when the gathering index of patterns was low, the presence of repeated elements appeared to play a crucial role in determining the perceived degree of order and disorder.

However, it is important to note that the human visual system is designed to process a retinal image that is two-dimensional and continuous. The data can be one-dimensional or discrete, but it is uncertain whether the visual system processes the data shown on a computer screen in such a way. Therefore, studying the cluster illusion visually would require using the 2D continuous space.

Evidences of humans' perceptual sensitivity to variations in the relative clustering or regularity in 2D dot patterns can be found in studies such as those exploring the perceived numerosity of those patterns. It has been shown that regular patterns are generally perceived as more numerous than random ones, and random patterns are perceived as more numerous than clustered ones [10, 11]. Moreover, Dry et al. [12] demonstrated that the degree of clustering, randomness and regularity in the 2D dot patterns can influence the ease with which people can solve TSP (Traveling Salesman Problem) and MST-P (Minimum Spanning Tree Problem). The results indicated that the human performance on those problems progressively decreased as the stimuli varied from clustered, through random, to regular [12].

However, whether humans can successfully discriminate between regular, random and clustered dot patterns is a different question. Many studies have shown that, although their perception of randomness is biased, nevertheless, humans can in fact perceive and respond to it. In his study, Preiss [13] showed that the human participants were capable of assessing the degree of clustering, randomness or regularity of 2D dot patterns. The results of the subjective ratings of randomness were remarkably close to the actual scores (based on the nearest neighbor statistic) that were used to measure the objective randomness of those patterns. This implied that the participants seemed not only to possess certain amount of knowledge about the average distance between the nearest neighbors within a given pattern, but also the ability to estimate the average expected distance between nearest neighbors in a truly-random pattern that has a similar number of dots [13].

Yamada et al.'s study [4] demonstrated that human observers are able to adapt to pattern randomness with the randomness aftereffect – after a prolonged exposure to a highly regular pattern, they tended to rate subsequent patterns as more random; and, conversely, if the initial pattern was highly random, they rated the following patterns as less random. The authors of the study suggest that this aftereffect may describe a mechanism present in the visual system that allows humans to distinguish randomness and regularity in the environment. Moreover, in a later study, Yamada [14] found some evidence to suggest that the visual perception of pattern randomness can be influenced by gender and age differences of the observers.

In two separate studies conducted with a time difference of 11 years, Griffiths and Tenenbaum [15, 16] attempted to construct Bayesian frameworks for describing how humans make judgments about randomness. The underlying idea behind their research was that subjective judgments of randomness are based on statistical inference, and that humans use prior knowledge to make predictions about the likelihood of outcomes. In both studies they used the same visual stimuli which consisted of 12 dot patterns generated with a mixture of Gaussian and Uniform distributions, with 4 varying parameters that determined the visual properties of the images: number of dots, proportion of dots within a cluster, spread and location of the cluster. While due to the simplicity of their earlier models the results allowed certain discrepancies [15], in their later study [16] the authors succeeded in creating a much simpler model which did an impressive job at capturing humans' intuitions behind distinguishing between random and non-random dot patterns (with a linear correlation of $r=0.951$ between the predictions and the human responses). However, there were still certain shortcomings of the model, as discussed by the authors, one of them being the fact that the model predicted a significant effect of the number of points within the dot patterns that was not present in the data, which could be either because humans' visual systems are less sensitive to this property of the data, or due to overlapping of points that made counting dots difficult [16].

To sum up, there is more than enough evidence to suggest that humans are, in fact, capable of perceiving randomness in the environment – in 2D patterns, specifically. This perception is, however, biased to some extent. Thus, a question arises: is it possible to coherently describe the distribution of subjective randomness?

Consider a segmentation of a given 2D dot pattern into a grid of regions with identical area size, similar to the procedure found in Clarke [5]. If the dots follow a point process (that is, if the positions of the dots are truly-random), the probability of a given number of dots in the segment is described by the Poisson distribution. If the positions of the dots are not truly-random but are subjectively-random, the dots are more evenly scattered in the 2D plane so that they do not form clusters. So, the probability of a given number of dots in the segment should be represented by a distribution that is expected to be different from the Poisson distribution.

In this study, I describe the aforementioned distribution of subjective randomness. I review the past research studies exploring subjective randomness, generate random dot patterns with the algorithms used for creating visual stimuli in their experiments, and analyze them to find potentially coherent statistical properties to describe the human perception of randomness. Based on those coherent properties, I suggest how a potential model characterizing the subjective-randomness can be developed, and discuss its consistency with other properties of the visual system.

## II. METHODS

### A. Algorithms for Generating 2D Dot Patterns

The analysis utilizes three algorithms that have been previously used to generate visual stimuli for behavioral experiments. The first of these algorithms was replicated from scratch; the second one was provided by Dry, M. (one of the authors of Dry et al. [12], which used the same visual stimuli for their experiment) and was then translated from MATLAB to Python with some modifications; the third one was provided by Sawada, T., who was the supervisor of the author of the study.

The following section provides an overview of each of these three algorithms. The methods for setting the parameters for those algorithms to generate patterns with the required distribution of dots will also be described. For each algorithm, the number of dots was varied between 49 (7×7), 100 (10×10) and 196 (14×14).

The first algorithm was used to generate visual stimuli in Yamada et al.'s [4] study. The experiment used patterns that were made up of 256 black dots, arranged in a 16×16 grid, each with a radius of 2 pixels. The background of the display was a white square with two dimensions of 272 pixels. The location of each dot was determined using a continuous uniform probability density function with a mean of $\mu$ and a range of ω in both the horizontal and vertical directions:

$$f(X) = \begin{cases} \frac{1}{\omega}, & \mu - \frac{\omega}{2} < X < \mu + \frac{\omega}{2} \\ 0, & Otherwise \end{cases} . \qquad (1)$$

Here, X denotes the position of each dot within the pattern, $\mu$ corresponds to a hypothetical dot position when the dots are perfectly aligned on a grid, and ω determines the level of physical randomness of the pattern.

The larger the parameter ω, the more physically random the pattern is. For the purposes of their study, the parameter was varied at seven levels, ranging from 2 to 14 pixels. In a later study, Yamada [15] used the same visual stimuli to test the effect of gender and age differences on the perception of randomness. In general, the results indicated an almost linear increasing trend of estimated subjective randomness from value 2 to 14 (that is, the patterns with ω =14 were perceived as the most random by the participants).

For the purposes of this study, three values of the parameter ω were used to generate the dot patterns: 2, 14 and 26. The values 2 and 14 correspond to patterns that were rated as least and most random, respectively, by the participants in Yamada's [15] study. The value of 26 was not tested in his study; however, it is used here to show that the distribution of dots becomes closer to that of true randomness as the value of ω increases. Fig. 1 shows patterns generated with three values of parameter ω (7; 14; 26) and three values of n (7; 10; 14).

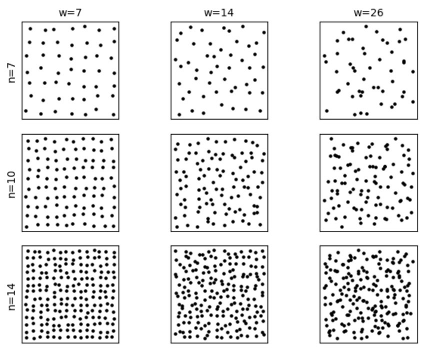



*Fig. 1*

The second algorithm was used for generating visual stimuli in Preiss [13] and Dry et al. [12]. The points are generated based on the nearest neighbor statistic R, which represents the ratio of the observed distance between the nearest neighboring points to the expected distance between nearest neighboring points under the assumption of complete spatial randomness.

The observed mean nearest neighbor distance for a set of *n* points is given by

$$r_0 = \frac{1}{n}\sum_{i \neq j}^{n} min\{u_{ij}\}, \qquad (2)$$

where $u_{ij}$ is the distance between the points *i* and *j*.

The nearest neighbor distance for *n* points in an area *A* under complete spatial randomness is given by the probability density function

$$p(d) = 2\pi\delta d e^{-\pi\delta d^2}, \qquad (3)$$

where $\delta = {}^{n}/_{A}$ is the mean number of points per unit area. So, the expectation of this distribution gives the mean nearest neighbor distance for a random process:

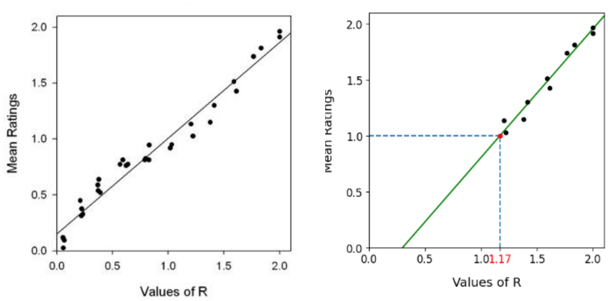
$$r_E = 0.5\sqrt{{}^{A}/_{n}}. \qquad (4)$$

The ratio of the observed and the expected mean nearest neighbor distances gives the nearest neighbor statistic:

$$R = \frac{r_0}{r_E} . \qquad (5)$$

Note that the closer the points are to being randomly distributed, the closer the values of $r_0$ and $r_E$ are, and so the closer to 1 the value of R gets. On the other hand, as points get more clustered, $r_0$ decreases, and the value for R becomes closer to 0. The value of R for perfect uniformity is 2.149; therefore, the closer the value of R is to 2.149, the more regularly spaced the points are.

As demonstrated in Preiss [13], the human perception of randomness in 2D dot patterns can be well captured by the value of the R-statistic. The relation between the rescaled mean ratings by the human observers and R-values was well described by a straight line, with an intercept close to 0 and a slope approaching 1 (see Fig. 2a).




*a)* *b)*

*Fig. 2*

Preiss's [13] algorithm is a composition of 3 separate algorithms for generating patterns with R<1, R=1, and R>1. As we can notice, the points in the plot can be classified into 3 groups based on their positions on the horizontal axis: below 1 (the five clusters on the left), around 1 (a single cluster at the center), and above 1 (the five clusters on the right). So, different algorithms were employed to generate the stimuli for these 3 groups. To estimate the R-value corresponding to the mean subjective rating of 1, the data points corresponding to the first two groups were neglected, and linear regression was performed on the points with an R-value greater than 1. The line of best fit indicated that the observers perceived patterns as the most random when the value of R was approximately 1.17 (see Fig. 2b).

To generate dot patterns for this study, the values of the mean neighbor statistic R were set to 1.00, 1.17 and 1.34, which were expected to generate patterns with respectively decreasing randomness (the value 1.00 corresponding to the most random dispersion of points). In Preiss's [13] study, the dot density was found to have no significant effect on categorizing the patterns as clustered, random or regular. Thus, the same values of R were used to generate dot patterns with three different values of n (as shown in Fig. 3).

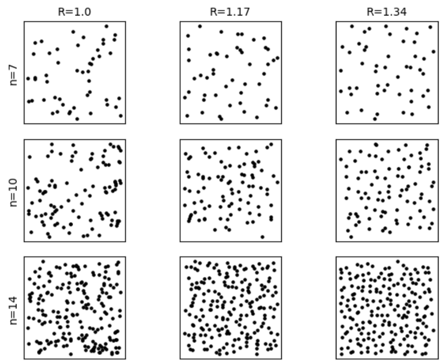



*Fig. 3*

The third algorithm used in this study for generating patterns was implemented to test human subjects in Sternik [17]. Similar to the one in Preiss [13], this algorithm also considers dot density using $r_E$ – that is, the expected nearest neighbor distance under a random process (see Eq. 4). During the experiment, the subjects were shown 2D scatterplots and were asked to obtain maximally random-looking patterns by adjusting the visual properties of the shown patterns (the adjustable parameter being the minimum distance of each point from its nearest neighboring point). To generate patterns for this study, I used the subjective scores of this parameter within one standard deviation of the mean of the human responses (see Fig. 4).

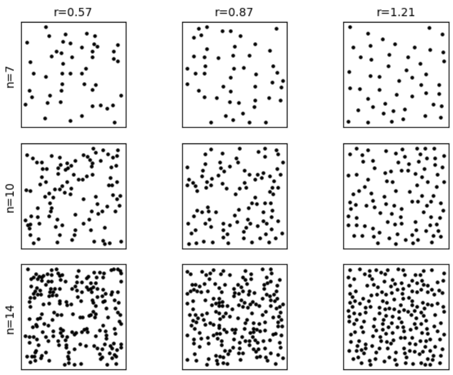



*Fig. 4*

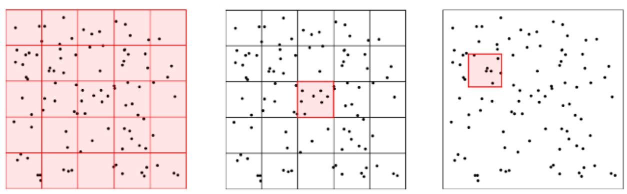
The algorithms used by both Preiss [13] and Yamada et al. [4] to produce dot patterns applied a technique of starting with a square grid as the initial placement for the points and "shuffling" them by adding noise to reach a certain value of the descriptive parameter. As a result, a grid bias could be observed in some of the later results, which could be more or less minimized by randomizing the position of the considered subregion. In case of Sternik's [17] algorithm, this bias was not observed, since the initial pattern was randomly generated.

### *B. The Distribution of Subjective Randomness*

To describe the distribution of points in a 2D dot pattern, I considered a segmentation procedure described in Clarke [5]. The 2D space was segmented into a grid of square regions with identical area size and the numbers of dots in the individual square regions were counted. Note that if the positions of the dots are truly-random, the probability of a given number of dots in the square region is represented by a Poisson distribution, the shape of which changes depending on the area size of each region. As the size of the region increases, the number of dots within that region increase, and the Poisson distribution becomes closer to the normal distribution. Conversely, if the segment size decreases, there are fewer dots in the segment, and the Poisson distribution takes on a more skewed shape. If the positions of the dots are subjectively-random rather than truly-random, they are more uniformly distributed, preventing clustering. So, this tendency should be reflected in a distribution describing subjective randomness that is expected to differ from the Poisson distribution.

To find a way of describing the properties of this distribution, I analyzed patterns that reflected the properties of subjective randomness and were generated using different algorithms. Using each one of the algorithms by Yamada et al. [4], Preiss [13], and Sternik [17] separately, $N=5000$ subjectively-random patterns were generated. Each pattern was divided into a grid of segments of different sizes, with $s$ segmentations per axis (2 to 10 segments per axis; so, the number of 2D regions varied from 4 to 100). Afterwards, the number of dots falling inside each subregion was counted to obtain a distribution characterizing the probability of a given number of dots within a single square region.

Three methods of counting the number of dots per segment were considered: (a) for each pattern, counting the frequencies of the number of dots in all $s^2$ segments and adding the results for all N patterns; (b) for each pattern, choosing one square segment close to the middle of the 2D segmented plane and adding the results for all N patterns; (c) for each pattern, choosing one square segment randomly within the pattern and adding the results for all N patterns (see Fig. 5). The former approach was the least computationally efficient (in the first



*a)* *b)* *c)*

*Fig. 5: The segmentation of a random 10×10 pattern (s=5)*

case, overall N•s$^2$ square regions were considered, while in case of the second and third methods only N); other than that, the results for the first two were generally similar. The third method was chosen based on the fact that it appeared to smooth down the grid bias present in the algorithms of Preiss [13] and Yamada et al. [4].

The results were summarized in histograms for each algorithm separately, and were compared to the histograms for truly-random patterns on each level of s – that is, the Poisson distribution. Basic statistical measures (mean, median, variance, skewness, kurtosis) were computed for the results produced with each algorithm. The change in the value of each measure as a function of the expected mean number of dots per subregion was considered. The plots were compared to the curves of the corresponding statistical measures for the Poisson distribution as functions of the mean (Tab. 1).

| Notation | Poisson($\lambda$) |
|---|---|
| Parameter | $\lambda > 0$ |
| Distribution | $k = 1, 2, 3, \dots,$ |
| PDF | $\frac{\lambda^k e^{-\lambda}}{k!}$ |
| CDF | $\sum_{i=1}^{k} \frac{\lambda^k e^{-\lambda}}{k!}$ |
| Mean | $\lambda$ |
| Variance | $\lambda$ |
| Standard Deviation | $\lambda^{\frac{1}{2}}$ |
| Skewness | $\lambda^{-\frac{1}{2}}$ |
| Kurtosis | $\lambda^{-1}$ |

*Tab. 1*

To fit a probability distribution to the obtained distributions of subjective randomness, a Poisson distribution can be scaled horizontally about the origin by a factor of k.

For a probability distribution given by the density function *P(X)*, the scaled probability distribution function is of the form:

$$P_k(X) = \frac{1}{k} P\left(\frac{X}{k}\right). \tag{6}$$

The mean and variance of the scaled distribution are given by

$$E[kX] = kE[X], \qquad Var(kX) = k^2 Var(X). \tag{7}$$

For a Poisson distribution with parameter $\lambda$, $\mu = \sigma^2 = \lambda$, where $\mu$ and $\sigma$ are the mean and the standard deviation, respectively.

Let $M$ and $S^2$ denote the estimated mean and variance of the distribution of subjective-randomness, respectively. Then, setting those to be equal to the expected mean and variance of the Poisson distribution, the parameter $\lambda$ of the Poisson distribution and the scaling factor k can be uniquely determined from the following equations:

$$k\lambda = M, \qquad k^2\lambda = S^2. \tag{8}$$

Thus, the distribution will be of the form:

$$d(X;\ k, \lambda) = \frac{1}{k} \frac{\lambda^{\frac{x}{k}} e^{-\lambda}}{\left(\frac{x}{k}\right)!}, \tag{9}$$

The density function can be generalized to the set of all non-negative real numbers by using the identity $\Gamma(n+1) = n!$, where $\Gamma(x)$ is the Gamma function, and $n$ is any non-negative integer. So, the PDF of the distribution is given by

$$d(X;\ k, \lambda) = \frac{1}{k} \frac{\lambda^{\frac{x}{k}} e^{-\lambda}}{\Gamma\left(\frac{x}{k} + 1\right)}, \tag{10}$$

$$k = \frac{S^2}{M}, \qquad \lambda = \frac{M^2}{S^2}. \tag{11}$$

### C. *The Pyramid Model*

How does the shape of the distribution change when the segment size changes? To answer this question, the concept of the image pyramid can be considered.

When an image is scaled to different sizes and its versions at different resolutions are "stacked" on one another in decreasing order, they constitute the layers of the square-base pyramid. The bottom layer represents the highest-resolution version of the image and the consequent layers represent higher stages of visual processing [18].

The images are represented on multiple levels of scale and resolution by the human visual system [19, 18]. It has been shown that the properties of the pyramid model capture human performance in cognitive tasks based on visual perception. Pyramid algorithms have been used to model human performance in problem solving for TSP, E-TSP, 15-puzzle and other optimization problems [20].

The assumtion was that the distribution for subjectively-randomness can be also formulated by making use of the pyramid model. The pyramid model makes the assumption of the same process applied across the levels. Changing the size of the subregion is like accessing different levels of the image pyramid of the 2D space. The dots in subjectively-random 2D patterns evenly scattered than in truly-random random patterns, which can be captured by the scale constant of the image pyramid.

## III. Results & Discussion

Consider the plots of the statistical measures (vertical axis) as functions of the expected theoretical number of dots per subregion under segmentations of size $s = 2,3, \dots 10$ (horizontal axis). Note that as s increases, the size of a unit subregion decreases, and is calculated by $n^2/s^2$ (where $n^2$ is the total number of dots within the image).

The mean and variance of the Poisson distribution are equal to the expected value λ. Therefore, the curves of the theoretical mean and variance against the expected value are represented by straight lines with slope 1. The skewness and kurtosis are both represented with decreasing functions (see Tab. 1). Clearly, as the size of the region increases, the number of dots within that region increases, and the shape of the Poisson distribution becomes closer to the bell-shape of the normal distribution – that is, the skewness and the kurtosis (describing the properties of symmetry and "tailedness") decrease, getting closer to 0.

The resulting plots of the statistical properties of the distributions, as compared to the properties of the Poisson distribution, exhibited some expected trends. The most noteworthy consistency across the three algorithms was that the variance obtained from any of the algorithms was markedly lower than the variance obtained from the Poisson process under the same procedure (see Fig. 6).

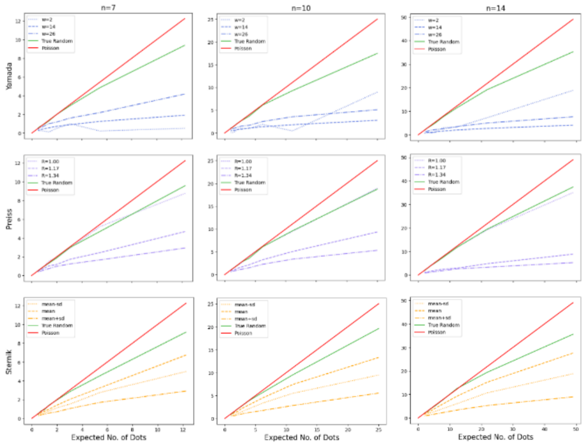


*Fig. 6: Estimated Subjective Variance as a Function of the Expected Mean*

For the scatter plots generated with Yamada's algorithm, the variance increases with the increase in the parameter ω (from 7, through 14, to 26) which was expected (as a larger value of the parameter creates patterns increasingly more "shuffled" from the initial grid placement). However, the variance for both 14 and 26 remain significantly below the curve of the Poisson's variance. The variance from Preiss' stimuli also behaves similarly. As noted earlier, the algorithm applied 3 branches of generating dot patterns (R<1, R=1 and R>1), so the curve corresponding to the parameter R=1.00 (created by random point generation) simply coincides with the estimated variance of the true-random patterns. As the R-statistic gets further from the value 1, the dots become more evenly spaced. So, the curve for the parameter R=1.17, which was estimated in his experiment to create the most subjectively-random images, is closer to the theoretical Poisson curve than that for R=1.34. The results for Sternik's algorithm are also consistent across the three values of n: the curves get closer to the theoretical variance from the values of the parameter one standard deviation below the estimated mean (r=1.21), through the estimated mean (r=0.87), to one standard deviation above the estimated mean (r=0.57) of the human responses. Fig. 7 shows the comparison of the three main parameters for each algorithm (estimated by their respective experiments to produce the most subjectively-random dot images) by estimated variance, across three values of $n$.

As can be noted, the variance is consistently below the curve of the theoretical Poisson variance, which increases linearly as a function of the expected number of dots. This indicates that the distribution of subjective-randomness is different from the Poisson distribution, which assumes independence of individual events from one another. So, the processes that generate patterns with visual properties reflecting maximal subjective randomness do not create independent points in the 2D space.

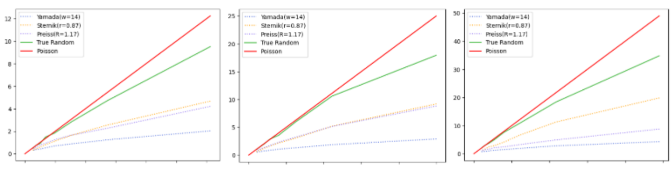


*Fig. 7*

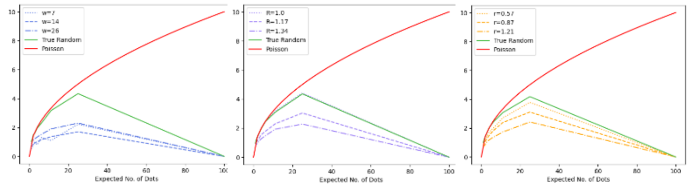


*Fig. 8*

On the other hand, the standard deviation of the Poisson distribution is not linear, as it is represented by $\lambda^{1/2}$, where $\lambda$ is the expected number of dots in a region. Since $\lambda = n^2/s^2$, when s=1 (i.e., no segmentation is applied), the expected number of dots within the segment is equal to the total number of dots within the whole pattern ($\lambda = n^2$). As $n$ is fixed (in this case – 7, 10 and 14), the measured variance in this case will be equal to 0, and so will the standard deviation. As shown in Fig. 8, the measured standard deviation of the distribution of dots from truly-random patterns shows a discrepancy from Poisson (SD=0 for $\lambda = n^2$), which can be attributed to the effect of segmentation. Therefore, the standard deviation as a function of the expected number of dots cannot be expressed as a linear function passing through the origin, which deviates from the prediction of the pyramid model. The theoretical SD of the pyramid model should be linear (with an intercept of 0), while the graphs for the truly-random patterns show a clear convex-upward trend. This non-linear trend was not tested in the past empirical studies, as they did not consider the effect of segmentation.

Therefore, some modifications are needed both for the Poisson distribution and the pyramid model, in order to be able to use them for modeling the distribution of subjective randomness and the change of that distribution against the change of the subregion area size. Using a scaled Poisson distribution could be one option, as the results of the fitted histograms for the three maximal-subjective-randomness parameters of the algorithms suggested that it could be a potentially working method. Further analysis and comparison of the scaled distributions across different algorithms and parameters could provide deeper insight into a coherent distribution modeling subjective randomness.

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